THE JOURNAL

OF THE

Indian Mathematical Society

5-SEP 1922

EDITED BY

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Hony. Joint Secretary

WITH THE CO-OPERATION OF

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and others.

VOL. XIII.

Madras:

PRINTED BY SRINIVASA VARADACHARI & CO.

4, MOUNT ROAD.

1921

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THE JOURNAL

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Indian Mathematical Society

Vol. XIV.] FEBRUARY 1922. [No. 1.

PROGRESS REPORT.

In reply to the circular dated 8th February 1922 about the constitution of the new Committee, about 70 members of the Society responded accepting the nominations of the retiring Committee and no alternative proposal was made by any member. Hence it is hereby announced that the Managing Committee is duly constituted as follows :—

Balak Ram, Esq., I.C.S. (*President*)

S. Narayana Aiyar, Esq., M.A. (*Treasurer*)

V. B. Naik, Esq., M.A. (*Librarian*)

R. P. Paranjpye, Esq., D.SC.

A. C. L. Wilkinson, Esq., M.A.

D. D. Kapadia, Esq., M.A.

M. T. Naraniengar, Esq., M.A.

P. V. Seshu Iyer, Esq., B.A.

E. B. Ross, Esq., M.A.

K. Ananda Rao, Esq., M.A.

S. V. Ramamurthy, Esq., I.C.S.

Pandit Hemraj, Esq., M.A.

2. The Committee has made the following changes in the office-bearers of the Society :—

Messrs. M. T. Naraniengar and P.V. Seshu Iyer are appointed Hon. Joint Secretaries, the former to do the duties of the Editor of the Journal

as before and the latter to do the duties of the Hon. Secretary of the Society in the place of Professor D. D. Kapadia.

Mr. K. Ananda Rao is appointed Joint Editor of the Journal in the place of Mr. P. V. Seshu Iyer appointed Hon. Secretary.

Mr. V. A. Apte, M.A., of the Fergusson College is appointed Hon. Asst. Librarian in the place of Mr. V. B. Naik appointed Hon. Librarian.

3. The following gentlemen have been admitted as members of the Society :—

(1) N. M. Shah, Esq., B.A. (Bom. & Cantab) Professor of Mathematics, New Poona College, Poona City.

(2) P. K. Kashiker, Esq., B.A. (Hon.), Teacher of Mathematics, Sangli City High School, Sangli (at concession rate)

(3) G. V. Telang, Esq., B. A, Senior Auditor, Local Fund Accounts (Secretariat, Bombay), Bagalkot District, Bijapur.

(4) S. R. U. Savoor, Esq., D.SC., Smith's Prizeman, Professor of Physics, Presidency College, Madras.

(5) T. V. Ramamurthy, Esq., B A., B.L., "New Place," Mylapore.

(6) T. B. Balagopal, Esq., B.A. (Hon.), 1 & 2, East Mada Street, Mylapore (at concession rate).

(7) V. A. Apte, Esq., M.A., Professor of Mathematics, Fergusson College, Poona.

(8) G. R. Narayana Aiyar, Esq., M.A., Lecturer in Mathematics, Ernakulam College, Ernakulam.

(9) S. Adinarayanan, Student, B.A. (Hon.), Presidency College, 39, Akbar Sahib Street, Triplicane (at concession rate).

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P. V. SESHU IYER,
Hony. Secretary.

FOCUS-ACNODAL CUBIC.

BY PANDIT HEMRAJ.

I. If a point on a curve of the 3rd degree be a focus, to find its equation referred to the focus as origin.

Suppose that the equation is

$$\alpha x + \beta y + ax^2 + 2hxy + by^2 + Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 = 0.$$

Now, $y = ix$ cuts the curve in points whose abscissæ are given by $x = 0$ and $\alpha + i\beta + (a - b + 2ih)x + (A + 3iB - 3C - iD)x^2 = 0.$

Hence $y = ix$ touches it, if

$$\left.\begin{aligned}(a-b)^2 - 4h^2 &= 4\{\alpha(A-3C) - \beta(3B-D)\};\\ 4h(a-b) &= 4\{\beta(A-3C) + \alpha(3B-D)\}.\end{aligned}\right\} \quad \ldots \quad (1)$$

Squaring and adding, we get

$$\{(a-b)^2 + 4h^2\}^2 = 16(\alpha^2 + \beta^2)\{(A-3C)^2 + (3B-D)^2\} \quad (2)$$

The abscissa of the point of contact is given by

$$x = -2\frac{(\alpha + i\beta)(a - b - 2ih)}{(a-b)^2 + 4h^2}.$$

If the point of contact be real, considering the values of both x and y, we get

$$\alpha(a-b) + 2h\beta = 0,$$
$$\beta(a-b) = 2h\alpha,$$

whence

$$\alpha^2 + \beta^2 = 0, \text{ or } (a-b)^2 + 4h^2 = 0.$$

Hence from (2) we get $h = 0,\ a = b,\ \alpha = 0,\ \beta = 0,$

or

$$a = b,\ h = 0,\ A = 3C,\ D = 3B.$$

Taking the 1st solution, we get the acnodal cubic

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 + a(x^2 + y^2) = 0 \quad \ldots \quad (3)$$

The 2nd solution gives the circular cubic

$$(x^2 + y^2)(Ax + Dy + a) + \alpha x + \beta y = 0, \quad \ldots \quad \ldots \quad (4)$$

which touches the line $y = ix$ at the circular point at infinity.

We call the cubic (4) a 'Focus Acnodal Cubic.'

II. Consider the cubic

$$F(x, y) \equiv \alpha x^2 + \beta y^2 + ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0.$$

The feet of the normals from the acnode to the curve lie on the cubic

$$x\frac{dF}{dy} = y\frac{dF}{dx},$$

i.e. $\quad \frac{2}{3}(\beta - \alpha)xy + bx^3 + (2c - a)x^2y + (d - 2b)xy^2 - cy^3 = 0.$

Four intersections of the two cubics coincide with the origin and there can be, therefore, drawn only five normals other than the normals at the node.

If $\beta = 0$, the feet lie on the nodal cubic whose one nodal tangent coincides with the cuspidal tangent of the original cubic. Hence five intersections coincide with the origin. Only four real or finite normals can be drawn from the cusp.

Eliminating y^3, we see that the feet lie on the conic

$$(ca - bd)\,x^2 + (3bc - 2cd + ad)\,xy$$
$$+ (3c^2 - d^2 + 2bd)\,y^2 + acx + \tfrac{2}{3}ady = 0.$$

If $\alpha = \beta$, the feet of the normals from the acnode lie on

$$bx^3 + (2c - a)\,x^2y + (d - 2b)\,xy^2 - cy^3 = 0. \quad \ldots \quad (5)$$

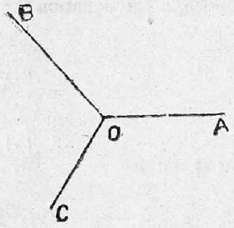
This represents three straight lines and is, therefore, the equation of three normals that can be drawn from the origin.

If m_1, m_2, m_3 be the tangents of the angles the normals make with the x-axis, then

$$\Sigma m_1 = \frac{d - 2b}{c}; \ \Sigma m_1 m_2 = \frac{a - 2c}{c}; \quad m_1 m_2 m_3 = \frac{b}{c}. \quad \ldots \quad (6)$$

III. Suppose that the normals from the focus-acnode are equally inclined to one another.

The equation of the cubic when a normal is taken as the x-axis will be of the form



$$ax^3 + 3cxy^2 + dy^3 + x^2 + y^2 = 0.$$

Normals are

$$y = 0, \ cy^2 - dyx + (a - 2c)\,x^2 = 0.$$

$$\therefore \ m_1 + m_2 = \frac{d}{c}, \ m_1 m_2 = \frac{a}{c} - 2 \ \ldots \quad (7)$$

But $\qquad m_1 = -m_2 = \sqrt{3}.$

$\therefore \qquad d = 0$ and $a + c = 0.$

$\therefore$ The equation of the curve takes the form

$$ax^3 - 3axy^2 + x^2 + y^2 = 0.$$

The feet A, B, C are $\left(-\frac{1}{a}, 0\right)$, $\left(\frac{1}{2a}, -\frac{\sqrt{3}}{2a}\right)$, $\left(\frac{1}{2a}, \frac{\sqrt{3}}{2a}\right)$.

Hence $\qquad$ OA = OB = OC.

IV. The foregoing results are deduced at once by considering the polar equation of the cubic, which may be written in the general form

$$r\,(\cos 3\theta + l \cos\theta + m \sin\theta) = a,$$

remembering that $\cos^3\theta = \dfrac{\cos 3\theta}{4} + \dfrac{3\cos\theta}{4},$

and $\sin^3\theta = \dfrac{3\sin\theta}{4} - \dfrac{\sin 3\theta}{4}$

and choosing the initial line suitably.

The normals from the origin are given by the condition

$$\frac{dr}{d\theta} = 0, \text{ or } \frac{du}{d\theta} = 0,$$

which is the same as

$$3\sin 3\theta + l\sin\theta - m\cos\theta = 0.$$

If $l = m = 0$, we have the special case of normals equally inclined to one another. The cubic in this case reduces to the well-known curve

$$r\cos 3\theta = a,$$

whose vertices form an equilateral triangle.

V. It is easy to see that the acnode is the triple focus of the cubic $r\cos 3\theta = a$. To find the single foci, we proceed as follows:—

Consider the line $r(\cos\theta + i\sin\theta) = \rho$, passing through the point $(\rho, 0)$. This line will touch the cubic provided the equation

$$\rho\cos 3\theta = a\,(\cos\theta + i\sin\theta)$$

has equal roots.

In other words, the point $(\rho, 0)$ is a single focus, if the equation

$$\cos 3\theta = \frac{a}{\rho}\,e^{i\theta} = \frac{a}{\rho}\,\lambda \text{ (say)},$$

has equal roots.

That is, if $\lambda^3 + \dfrac{1}{\lambda^3} = 2\,\dfrac{a}{\rho}\,\lambda$

has equal roots.

$\therefore \quad \lambda^6 - 2\,\dfrac{9}{\rho}\,\lambda^4 + 1 = 0$ must have equal roots.

Hence the discriminant $27 - 32\,\dfrac{a^3}{\rho^3} = 0.$

$$\therefore \quad \rho = \tfrac{2}{3}\,.\,2^{\frac{2}{3}}\,.\,a.$$

The symmetry of the figure shows that there are two more single foci at points $(\rho, 120^\circ)$, $(\rho, 240^\circ)$.

VI. Further, if r_1, r_2, r_3 be the distances of any point P on $r\cos 3\theta = a$ from the single foci F_1, F_2, F_3, we have

$$\left.\begin{aligned} r_1^2 &= r^2 + \rho^2 - 2r\rho\cos\theta, \\ r_2^2 &= r^2 + \rho^2 - 2r\rho\cos(\theta + 120^\circ), \\ r_3^2 &= r^2 + \rho^2 - 2r\rho\cos(\theta + 240^\circ); \end{aligned}\right\}$$

and therefore r_1^2, r_2^2, r_3^2 are the roots of the cubic equation

$$4\left(\frac{x - (r^2 + \rho^2)}{2r\rho}\right)^3 - 3\left(\frac{x - (r^2 + \rho^2)}{2r\rho}\right) + \frac{a}{r} = 0;$$

i.e. $$x^3 - 3x^2(r^2 + \rho^2) + 3x(r^4 + \rho^4 + r^2\rho^2) - r^6 - \rho^6 + 2ar^3\rho^3 = 0.$$

VII. Again, the equation of the focus acnodal cubic is

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = x^2 + y^2.$$

The axes cut the curve in points $\left(\frac{1}{a}, 0\right)$ and $\left(0, \frac{1}{d}\right)$.

The tangents at these points are

$$ax + 3by = 1,\ 3cx + dy = 1.$$

Their intersection evidently lies on the cubic.

Since any two perpendicular lines through the acnode may be taken as axes, we have :—

(1) Tangents at the extremities of any two perpendicular lines through the focus-acnode of a cubic intersect on the cubic.

It is easy to show that

(2) The fourth harmonic conjugate of any normal from the focus-acnode, with respect to the other two is perpendicular to the fourth harmonic conjugate of an asymptote of its Hessian with respect to the other two asymptotes.

(3) The fourth harmonic conjugate to the three normals of a focus-acnodal cubic, of which two are perpendicular to each other, is parallel to the line through the feet of the perpendicular normals.

Again, the equation of the cubic when one normal is taken as x-axis becomes $$ax^3 + 3cxy^2 + dy^3 = x^2 + y^2.$$

Normal at any point (x, y) passes through (h, k), if

$$(h - x)(2cxy + dy^2 - \tfrac{2}{3}y) = (k - y)(ax^2 + cy^2 - \tfrac{2}{3}x)$$

If $k = 0$, $y = 0$ and $(2c - a)x^2 + dxy - cy^2 = 2chx + dhy - \frac{2}{3}h.$

Hence we have

(4) If from any point on a real normal from the focus-acnodal are drawn normals to the focus-acnodal cubic, their feet other than the foot of the normal through the acnode, lie on a conic having its asymptotes parallel to the other two normals from the acnode.

Further we notice that the conic will be a line-pair, if

$$\tfrac{3}{2}\,h = \frac{d^2 + 4c\,(2c - a)}{4c^3 + ad^2}.$$

It follows that

(5) There exist three line-pairs such that the normals at the intersections of each pair with a focus-acnodal cubic intersect at a point lying on a normal from the acnode.

(6) The asymptotes of the cubic which passes through the feet of the normals from any point to a focus-acnodal cubic, are parallel to the normals from the acnode.

VIII. If two of the normals from the focus acnode be perpendicular to each other, then it is easy to see that

$$b^2 + c^2 = ac + bd.$$

Referring to these normals as axes, the equation n takes the simple form $$ax^3 + dy^3 = x^2 + y^2.$$

The normal at the point of intersection of the tangents at the feet of the perpendicular normals is $\quad dy = ax.$

Hence we have

(7) If there exist two perpendicular normals from the focus-acnode of a cubic, the tangents at their feet intersect at the foot of the third normal.

Again, the equation to BC, where B, C are the feet of perpendicular normals and A that of the third, is $ax + dy = 1$.

Tangent at A is $$\frac{x}{a} + \frac{y}{d} = \frac{1}{a^2} + \frac{1}{d^2}.$$

Let BC and the tangent at A intersect the curve again in D and E;

D is $\left(\dfrac{a}{a^2 - d^2},\ \dfrac{d}{d^2 - a^2}\right)$ and the tangent thereat is

$$a(a^2 + 2d^2)\,x + d(2a^2 - d^2)\,y = d^2 + a^2\,;$$

E is $\left(\dfrac{a^2 + d^2}{a\,(d^2 - a^2)},\ \dfrac{a^2 + d^2}{d\,(a^2 - d^2)}\right)$ which lies on the tangent at D.

Hence we have

(8) The line joining the feet of the perpendicular normals from the focus-acnode of a cubic and the tangent at the foot of the 3rd normal meet the curve in D and E. ED touches the curve at D.

This is a particular case of a general theorem on cubics. It can be proved geometrically also.

Again, the equation of the circle through the feet of normals is

$$x^2 + y^2 = \frac{x}{a} + \frac{y}{d}.$$

It follows that

(9) The circle through the focus-acnode and the feet of the perpendicular normals therefrom to the cubic touches the curve at the foot of the 3rd normal.

Again, if B, C be the feet of perpendicular normals and A that of the 3rd, and if ρ_2, ρ_3, ρ_1 be the radii of curvature, then it is easy to see that

$$\rho_2{}^2 + \rho_3{}^2 = 4\rho_1{}^2.$$

(10) The centres of curvature at the feet of the normals (two of them being perpendicular) from the focus-acnode lie on a straight line which is parallel to the line of flexes; and the tangents and normals at the finite flexes meet respectively on the 3rd normal.

If we put $u = y' - ix'$ and $a = \frac{1}{\alpha}$, $d = \frac{1}{\delta}$, the equation giving the foci is $u^3 - 2(\delta - i\alpha)u^2 + (\delta^2 - \alpha^2 - \frac{10}{3} i\alpha\delta)u + \frac{32}{27}\alpha\delta(\alpha + i\delta) = 0.$

Put $u = z + \frac{2}{3}(\delta - i\alpha)$ and we have

$$z^3 - \tfrac{1}{3}(\delta + i\alpha)^2 z + \tfrac{2}{27}(\delta + i\alpha)(\delta^2 + \alpha^2) = 0.$$

$$\therefore \qquad z_1 + z_2 + z_3 = 0.$$

i.e. $$x_1 + x_2 + x_3 = 2\alpha, \; y_1 + y_2 + y_3 = 2\delta.$$

Hence we have

(11) The centre of gravity of the triangle formed by the single foci of a focus-acnodal cubic having two perpendicular normals from the acnode lies on the 3rd normal from the acnode.

IX. If we denote by F and H a focus-acnodal cubic and its Hessian, then

$$F \equiv ax^3 + 3bx^2y + 3cxy^2 + dy^3 + x^2 + y^2 = 0,$$

$$H \equiv 3cx^3 + 3(d-2b)x^2y + 3(a-2c)xy^2 + 3by^3 + x^2y^2 = 0\ ;$$

and the quation $3F + H = 0$ gives

$$3(a+c)x + 3(b+d)y + 4 = 0,$$

which is the line of flexes.

Cor. If two of the normals be perpendicular to each other, the line of flexes is

$$3ax + 3dy + 4 = 0.$$

X. If two normals be perpendicular to each other and they be taken as axes, $b=0$, $c=0$, and $F-H = ax^3 - 3dx^2y - 3ax^2y^2 + dy^3 = 0.$

This represents the lines through the acnode and flexes.

[If one of the lines be $y = mx$, we have

$$\frac{a}{d} = \frac{3m-m^3}{1-3m^2} = \tan 3\theta.$$

(12) One flex divides the line joining the other two in the ratio of their distances from the acnode.]

XI. Again if (h, k) be a point on the Hessian, then

$$h^2(1 + 3ch + \overline{3d-2b}k) + k^2(1 + \overline{3a-2c}h + 3bk) = 0, \qquad \ldots (\alpha$$

and the polar conic of (h, k) is

$$h(2x + 3ax^2 + 6bxy + 3cy^2) + k(2y + 3bx^2 + 6cxy + 3dy^2) + x^2 + y^2 = 0.$$

If ξ, η be the reciprocals of the intercepts made by the polar conic on the axes, then

$$(2\xi + 3a)h + 3bk + 1 = 0\ ;\ 3ch + (2\eta + 3d)k + 1 = 0.$$

$$\therefore\ h = \frac{3b-3d-2\eta}{(2\xi+3a)(2\eta+3d)-9bc},\ k = \frac{3c-3a-2\xi}{(2\xi+3a)(2\eta+3d)-9bc} \ldots\ldots(\beta)$$

Since the polar conic of every nodal cubic passes through the node, the polar conic of (h, k) is the line $x\xi + y\eta = 1$ and a line through the origin.

The envelope of the former line is the Cayleyan.

Hence eliminating h, k between (α) and (β), we get the tangential equation of the Cayleyan, *viz.*

$$2(\xi^2 + \eta^2) + 3(a+c)\xi + 3(b+d)\eta + 9(bd + ac - b^2 - c^2) = 0.$$

Hence the Cayleyan is a central conic.

The Cartesian equation of the Cayleyan is

$$9\{(d+b)^2-8(bd+ca-b^2-c^2)\}x^2-18(a+c)(b+d)xy+9\{(a+c)^2$$
$$-8(bd+ca-b^2-c^2)\}y^2-24(a+c)x-24(b+d)y=16.$$

The tangents from the acnode to the Cayleyan satisfy the conditions for being a circle.

Hence the acnode is its focus.

Any line through the centre of the conic is

$$3[(d+b)^2-\lambda]x-3(a+c)(b+d)y-4(a+c)$$
$$+\mu\{3(a+c)(b+d)x-3[(a+c)^2-\lambda]y+4(b+d)\}=0,$$

where $\lambda=8(bd+ca-b^2-c^2)$.

It passes through the focus if $\mu=\dfrac{a+c}{b+d}$.

Hence the equation of the major axis of the conic is

$$(b+d)x=(a+c)y.$$

(13) The Cayleyan of the focus-acnodal cubic is a central conic whose major axis is perpendicular to the line of flexes.

Cor. If the normals be equally inclined to one another, the Cayleyan becomes the circle $(x^2+y^2)(a^2+d^2)=\frac{1}{9}$.

Cor. If two of the normals be perpendicular to each other, the Cayleyan is the parabola $2(\xi^2+\eta^2)+3(a\xi+d\eta)=0$;

that is $(dx-ay)^2=\frac{8}{3}(ax+dy+\frac{2}{3})$.

(14) The Cayleyan of a focus-acnodal cubic is a parabola whose axis is perpendicular to the lines of flexes and the line of the feet of two perpendicular focal normals and passes through the acnode and whose tangent at the vertex passes through the intersect of two of the asymptotes of the Hessian and whose focus is the acnode.

Further we notice that

(15) The feet of the perpendiculars from the acnode upon the inflexional tangents of the focus-acnodal cubic lie in general on a circle which becomes a straight line when two of the normals are perpendicular to each other and in the latter case the circumcircle of the triangle formed by the inflexional tangents passes through the acnode.

XII. If the perpendicular focal normals be equal in magnitude, the equation takes the form $x^3 + y^3 = a(x^2 + y^2)$.

Now there exist three line-pairs such that the normals at the intersections of each pair with the curve intersect at a point lying on a normal from the acnode.

Hence at the intersections of $x = \frac{2}{3}a,\ y = \frac{2}{3}a$;

$$x = \tfrac{2}{3}a,\ x - y = -\tfrac{2}{3}a\ ;\ y = \tfrac{2}{3}a,\ x - y = \tfrac{2}{3}a,$$

with the cubic $x^3 + y^3 = a(x^2 + y^2)$, the normals meet at

$$(\tfrac{2}{3}a,\ \tfrac{2}{3}a)\ ;\ (\tfrac{2}{3}a,\ 0),\ (0,\ \tfrac{2}{3}a)$$

The real asymptote is $x + y = \frac{2}{3}a$

The line of flexes is $x + y = \frac{4}{3}a$

The finite flexes are $\left[\frac{2}{3\sqrt{3}}(\sqrt{3} \pm 1)a,\ \frac{2}{3\sqrt{3}}(\sqrt{3} \mp 1)a\right]$

and therefore the lines joining the origin and flexes make with the x-axis angles of 75°, 15°

The points S and T are

$$\left[\frac{2}{3}a,\ \frac{a}{3}\left\{(\sqrt{2}+1)^{2/3} + (\sqrt{2}-1)^{2/3}\right\} + \frac{a}{3}\right],$$

and

$$\left[\frac{a}{3} + \frac{a}{3}\left\{(\sqrt{2}+1)^{2/3} + (\sqrt{2}-1)^{2/3}\right\},\ \frac{2}{3}a\right].$$

The line DK $(x - y = -\frac{2}{3}a)$ meets the curve at Q, whose co-ordinates are

$$[\tfrac{1}{3}2^{\frac{1}{3}}a,\ \tfrac{1}{3}(2^{\frac{1}{3}} + 2)a].$$

The line EH $(x - y = \frac{2}{3}a)$ meets it at P $[\frac{1}{3}(2^{\frac{1}{3}} + 2)a,\ \frac{1}{3}2^{\frac{1}{3}}a]$.

$$\therefore \qquad ON = PM = 2^{\frac{1}{3}}\,BH = 2^{\frac{1}{3}}CK = HM = KL.$$

$$\therefore \qquad ON^3 = 2\,BH^3\ ;\ PM^3 = 2\,CK^3.$$

Also we have $KQ^2 = HP^2 = 2^{-\frac{1}{3}}OK^2 = 2^{-\frac{1}{3}}OH^2$.

(16) Hence the duplication of a cube can be effected by means of this curve.

(17) HB, HS, HQ are normals at B, S, Q and KC, KT, KP are normals at C, T, P.

(18) The line joining the points of intersections of the tangents at Q, S and P, T is parallel to the line of flexes.

(19) The angle between OV and HQ is 45°.

(20) OP and OQ are equally inclined to the perpendicular normals.

Further, we notice that if $\lambda = \frac{\sqrt{2}}{3}a(\sqrt{2}-1)^{\frac{1}{3}}$; $\mu = \frac{\sqrt{2}}{3}a(\sqrt{2}+1)^{\frac{1}{3}}$,

the three single foci are

$$x_1 = \frac{2}{3}a - (\lambda - \mu)\sin\frac{3\pi}{4},\ y_1 = \frac{2}{3}a + (\lambda - \mu)\cos\frac{3\pi}{4};$$

$$x_2 = \frac{2}{3}a - \left(\lambda\sin\frac{\pi}{12} + \mu\cos\frac{\pi}{12}\right),\ y_2 = \frac{2}{3}a + \left(\lambda\cos\frac{\pi}{12} + \mu\sin\frac{\pi}{12}\right);$$

$$x_3 = \frac{2}{3}a + \left(\lambda\cos\frac{\pi}{12} + \mu\sin\frac{\pi}{12}\right),\ y_3 = \frac{2}{3}a - \left(\lambda\sin\frac{\pi}{12} + \mu\cos\frac{\pi}{12}\right).$$

Any point on the curve is

$$x = a\frac{(1+t^2)}{(1-t^2)},\ y = a\frac{t(1+t^2)}{(1-t^2)}.$$

Three points are collinear if $\Sigma t_1 + \Sigma t_1 t_2 = 1 + t_1 t_2 t_3$.

If $t_1 = t_2 = t_3$, the flexes are given by $t^3 - 3t^2 - 3t + 1 = 0$, which shows that one flex is at infinity.

Now we shall find the area bounded by the curve $x^3 + y^3 = x^2 + y^2$ and its asymptote.

Turn the axes through $\frac{\pi}{4}$ and the equation takes the form

$$x^3 + 3xy^2 = \sqrt{2}(x^2 + y^2).$$

$$\therefore\ \tfrac{1}{2}\text{ area} = \int_{\frac{1}{3}\sqrt{2}}^{\sqrt{2}} \left(\frac{\sqrt{2} - x}{3x - \sqrt{2}}\right)^{\frac{1}{2}} x\,dx = \frac{\pi}{3\sqrt{3}}.$$

Shift the origin to O′ or $\left(\frac{\sqrt{2}}{3}, 0\right)$ and the equation takes the form

$$3xy^2 = \frac{4\sqrt{2}}{27} + \tfrac{2}{3}x - x^3.$$

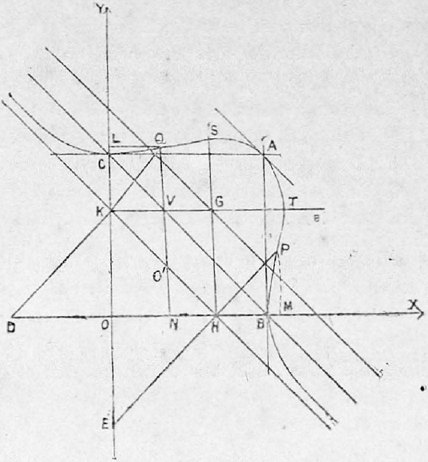
$$\therefore\ \tfrac{1}{2}\text{ area} = \frac{2^{\frac{3}{4}}}{3}\int_0^{\frac{2\sqrt{2}}{3}} \left(\frac{\sqrt{2}}{3} + x\right) x^{-\frac{1}{2}} \left(1 - \frac{3x}{2\sqrt{2}}\right)^{\frac{1}{2}} dx$$

$$\left[= \frac{2^{\frac{3}{4}}}{3}\int\left(\frac{\sqrt{2}}{3} + x\right)x^{-\frac{1}{2}}\left\{1 - \tfrac{1}{2}\frac{3x}{2\sqrt{2}} - \frac{1}{2^2!\,2}\left(\frac{3x}{2\sqrt{2}}\right)^2 - \dots\right\}dx,\right.$$

which gives

$$(21)\ \frac{3\pi}{8}=\left[1-\frac{1}{2}\cdot\tfrac{1}{3}-\frac{1}{2^2!\,2}\cdot\tfrac{1}{5}-\frac{1\cdot3}{2^3!\,3}\cdot\tfrac{1}{7}-\frac{1\cdot3\cdot5}{2^4!\,4}\cdot\tfrac{1}{9}-\dots\right]$$

$$+2\left[\tfrac{1}{3}-\frac{1}{2}\cdot\tfrac{1}{5}-\frac{1}{2^2!\,2}\cdot\tfrac{1}{7}-\frac{1\cdot3}{2^3!\,3}\cdot\tfrac{1}{9}-\frac{1\cdot3\cdot5}{2^4!\,4}\cdot\tfrac{1}{11}-\dots\right].$$



$$OB = OC = a;$$
$$OH = OK = \tfrac{2}{3}a.$$

XIII. If two normals from the acnode coincide, the equation of the curve takes the form $x^3=a\,(x^2+y^2)$, or $r\cos^3\theta = a$.

The line of flexes is $x=\frac{4}{3}a$. One flex is at infinity and the finite flexes are

$$x=\frac{4}{3}a,\ y=\pm\frac{4}{3\sqrt{3}}a.$$

Any point on it is $x=a\,(1+t^2)$; $y=a\,t(1+t^2)$.

ON A PARTICULAR TYPE OF SIMILAR TRIANGLES.

By F. H. V. Gulasekharan, M A.

If E is any straight line parallel to the Euler line of a triangle ABC, and if E_1, E_2, E_3 are the Euler lines of the triangles formed by (AB, AC, E), (BC, BA, E), (CA, CB, E), Professor Wilkinson states in Q 1068, *J. I. M. S.* certain interesting results connected with the triangle formed by E_1, E_2, E_3. The first part of his theorem, *viz* :—that E_1, E_2, E_3 are parallel to BC, CA, AB respectively—is an immediate inference of the extended form of Q. 1009, as shown in solution to Q. 1009. The remaining results are particular cases of the following general theorem :—

Theorem :—ABC is a given triangle and L *any given* straight line through its centroid ; *any* straight line L′ parallel to L cuts the sides BC, CA, AB at α, β, γ respectively ; G_1, G_2, G_3, are the centroids of the triangles $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ respectively. Through G_1, G_2, G_3 respectively straight lines B′C′, C′A′, A′B′ are drawn parallel to BC, CA, AB respectively.

Then

(1) the triangles ABC, A′B′C′ are equal in all respects.

(2) the loci of the vertices of the triangle A′B′C′, as L′ varies, are three straight lines parallel to the axis of the parabola which touches the sides of the triangle ABC and the given line L.

(3) if L′ passes through C, straight lines B′C′, C′A′, A′B′ and L are concurrent.

(4) The locus of the centre of homology of the triangles ABC, A′B′C′ is a straight line parallel to the axis of the parabola defined in (2).

Let us take ABC for the triangle of reference and use *areal* co-ordinates. The following lemmas will be found useful in deducing the above theorem :—

Lemma (i) If P, Q, R are any three points whose *absolute* areal co-ordinates are (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3), respectively ; and if through P, Q, R straight lines B′C′, C′A′, A′B′ are drawn parallel to the sides BC, CA, AB respectively of the triangle of reference,

then $$\frac{\triangle A'B'C'}{\triangle ABC} = (X_1 + Y_2 + Z_3 - 1)^2$$

Proof. The equations to B′C′, C′A′, A′B′ are $x = X_1 (x+y+z)$, $y = Y_2 (x+y+z)$, $z = Z_3 (x+y+z)$ respectively.

$$\therefore \quad \frac{\triangle A'B'C'}{\triangle ABC} = X_1^2 Y_2^2 Z_3^2 \begin{vmatrix} 1-\dfrac{1}{X_1}, & 1, & 1 \\ 1, & 1-\dfrac{1}{Y_2}, & 1 \\ 1, & 1, & 1-\dfrac{1}{Z_3} \end{vmatrix}^2$$

$$= (X_1 + Y_2 + Z_3 - 1)^2.$$

This result may also be readily seen geometrically.]

Lemma (ii). If (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3) be the *absolute* co-ordinates of the centroids of the triangles formed by (AB, AC, L′), (BC, BA, L′), (CA, CB, L′), where L′ is any straight line, then

$$X_1 + Y_2 + Z_3 = 2.$$

Proof:—Let L′ be the straight line $l'x + m'y + n'z = 0$, and suppose it cuts BC at α, CA at β, and AB at γ.

Then, the *absolute* co-ordinates of α are

$$\left(0, \frac{n'}{n'-m'}, \frac{m'}{m'-n'}\right);$$

similarly for the points β, γ.

Hence
$$3X_1 = 1 + \frac{m'}{m'-l'} + \frac{n'}{n'-l'};$$
$$3Y_2 = 1 + \frac{n'}{n'-m'} + \frac{l'}{l'-m'};$$
$$3Z_3 = 1 + \frac{l'}{l'-n'} + \frac{m'}{m'-n'}.$$

Hence $X_1 + Y_2 + Z_3 = 2$.

Lemma (iii). If straight lines are drawn through the vertices of ABC, the triangle of reference, parallel to the axis of the parabola touching the sides of ABC and the straight line $lx + my + nz = 0$, the equations to these straight lines are

$$\frac{y}{m(n-l)} = \frac{z}{n(l-m)}; \quad \frac{z}{n(l-m)} = \frac{x}{l(m-n)};$$
$$\frac{x}{l(m-n)} = \frac{y}{m(n-l)} \quad \text{respectively.}$$

The proof is simple and depends on the property that the join of A to the middle point of its chord of contact is parallel to the axis of the parabola.

Making now the points P,Q,R of *Lemma* (i) coincide with the centroids of *Lemma* (ii), and the straight line $lx+my+nz=0$ of *Lemma* (iii) coincide with the straight line L drawn through the centroid of ABC parallel to the line L′ of *Lemma* (ii), we shall then get all the data of the theorem.

Hence $\dfrac{\triangle A'B'C'}{\triangle ABC} = (X_1 + Y_2 + Z_3 - 1)^2$ from *Lemma* (i)

But $X_1 + Y_2 + Z_3 = 2$ from *Lemma* (ii)

∴ $\triangle A'B'C' = \triangle ABC.$

But △s A′B′C′, ABC are similar (*Data*)

∴ $\triangle A'B'C' \equiv \triangle ABC$

which proves the first part.

Again, since L passes through (1, 1, 1), and is parallel to L′,

$$l + m + n = 0;$$

and, $l' : m' : n' :: l + \lambda : m + \lambda : n + \lambda$, where λ varies with L′.

Now the equation to A′B′ is

$$3z = \left[1 + \frac{l+\lambda}{l-n} + \frac{m+\lambda}{m-n}\right](x + y + z).$$

Again if x_1, y_2, z_3 be the values of X_1, Y_2, Z_3 respectively, when L′ coincides with L, remembering that $l + m + n = o$, the equation to A′ B′ reduces to the form,

$$z - z_3(x + y + z) = \frac{\lambda n}{(m-n)(n-l)} \quad (x + y + z)$$

since $l + m + n = 0$.

Similarly the equation to A′C′ is of the form

$$y - y_2(x + y + z) = \frac{\lambda m}{(m-n)(l-m)}(x + y + z).$$

Hence eliminating λ between the last two equations, the locus of A′, as L′ varies, is the straight line

$$\frac{y}{m(n-l)} - \frac{z}{n(l-m)} = \left[\frac{y_2}{m(n-l)} - \frac{z_3}{n(l-m)}\right](x + y + z),$$

the form of which shows that the locus is parallel to the straight line

$$\frac{y}{m(n-l)} = \frac{z}{n(l-m)}$$

which by *Lemma* (iii) is parallel to the axis of the parabola touching L and the sides of the triangle ABC.

Similarly for the locus of B′ and C′.

Now to prove the third part, we notice that if L′ passes through C, $n' = 0$, or $\lambda = -n$.

Hence, remembering that $l + m + n = 0$, the equations to B′ C′, C′ A′

are $$x = \frac{m}{m-l}(x + y + z),$$

and $$y = \frac{l}{l-m}(x + y + z) \text{ respectively.}$$

Hence C′ lies on the straight line $lx + my = 0$, which passes through the intersection of AB and L.

Hence B′C′, C′A′, AB and L are concurrent.

(*d*) *To prove* § 1 (4)

The centre of homology of the triangles ABC′, A′B′C′ is given by the

point $$\frac{x}{X_1} = \frac{y}{Y_2} = \frac{z}{Z_3}.$$

By Lenma (ii) $X_1 + Y + Z_3 = 2.$

$$\therefore \quad \frac{x}{X_2} = \frac{y}{Y_2} = \frac{z}{Z_3} = \frac{1}{2}(x + y + z).$$

Now, if as before x_1, y_2, z_3 be the values of X_1, Y_2, Z_3 when L′ coincides with L, by methods similar to those adopted in the course of the proof in § 4. (*b*),

$$\frac{x}{X^1} = \frac{3x}{3x_1{}_3^2 - 3\lambda l/(n-l)(l-m)}$$

$$= \frac{x/l(m-n)}{x_1/l(m-n) + \lambda/(m-n)\ (n-l)(l-m)}.$$

$$\therefore \quad \frac{x/l(m-n) - y/m(n-l)}{x_1/l(m-n) - y/_2m(n-l)} = \frac{1}{2}(x + y + z), \text{ the form of}$$

which shows that the locus of the centre of homology is a straight line parallel to the axis of the parabola of *Lemma* (iii).

The following generalisations are interesting :—

(*a*) L , L_2, L_3 ...., L_n are any n straight lines at random ; L_r cuts the sides BC, CA, AB of a triangle ABC at the points α_r, β_r, γ_r respectively ; g_1, g_2, g^3, g_1', g_2', g_3', g_1'', g_2'', g_3'' are the centres of mean position of the sets of points $(A\beta_1\gamma_1\beta_2\gamma_2 .... \beta_n\gamma_n)$,

$(B\gamma_1\alpha_1\gamma_2\alpha_2 \ldots\ldots \gamma_n\alpha_n)$, $(C\alpha_1\beta_1\alpha_2\beta_2 \ldots\ldots \alpha_n\beta_n)$,
$(BC\beta_1\gamma_1 \ldots\ldots\ldots \beta_n\gamma_n)$, $(CA\gamma_1\alpha_1 \ldots\ldots\ldots \gamma_n\alpha_n)$,
$(AB\alpha_1\beta_1 \ldots\ldots\ldots \alpha_n\beta_n)$, $(\beta_1\gamma_1\beta_2\gamma_2 \ldots\ldots \beta_n\gamma_n)$,
$(\gamma_1\alpha_1\gamma_2\alpha_2 \ldots\ldots\ldots \gamma_n\alpha_n)$, $(\alpha_1\beta_1\alpha_2\beta_2 \ldots\ldots\ldots \alpha_n\beta_n)$ respectively;

straight lines B_1C_1, B_2C_2, B_3C_3 are drawn through g_1, g_1', g_1'' respectively parallel to BC; C_1A_1, C_2A_2, C_3A_3 through g_2, g_2', g_2'' respectively parallel to CA; A_1B_1, A_2B_2, A_3B_3 through g_3, g_3', g_3'' parallel to AB.

Then (i) $\dfrac{\Delta A_1B_1C_1}{\Delta ABC} = \left(\dfrac{n+2}{2n+1}\right)^2$;

(ii) $\dfrac{\Delta A_2B_2C_2}{\Delta ABC} = \left(\dfrac{n-2}{2n+2}\right)^2$;

(iii) $\dfrac{\Delta A_3B_3C_3}{\Delta ABC} = \dfrac{1}{4}$, *which is independent of n.*

Hence *in case* (iii) $\Delta A_3B_3C_3$ *is equal in all respects to the medial triangle of* ABC.

(*b*) If in (*a*) above, the several straight lines L_r be parallel to a given straight line L drawn through the centroid of ABC, the loci of the vertices of the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ and the loci of the centres of homology of the pairs of triangles (ABC, $A_1B_1C_1$), (ABC, $A_2B_2C_2$), (ABC, $A_3B_3C_3$) are fixed straight lines parallel to the axis of the parabola touching the given straight line L and the sides of the triangle ABC.

(*c*) If, in (*a*) (ii) above, $n = 2$, the points A′, B′, C′ are coincident.

(*d*) As a particular case of $\Delta A_3B_3C_3$, the following theorem may be stated :—

If any straight line cuts the sides BC, CA, AB of a triangle ABC at the points D, E, F respectively, and straight lines are drawn through the middle points of EF, FD, DA respectively parallel to BC, CA, AB respectively:

(i) *the triangle so formed is equal in all respects to the medial triangle of ABC;*

(ii) *as DEF moves parallel to itself, the loci of the vertices of the triangle so formed and the locus of the centre of homology of that triangle and ABC are fixed straight lines parallel to the axis of the parabola touching the sides of the triangle ABC and the straight line through the centroid of ABC parallel to DEF.*

SHORT NOTES

On Question 1160.

(N. B. MITRA). The problem proposed to be solved in this note is:— To find a nnmber N consisting of n digits in the scale r such that N^2 ends with the same n digits in the same order.

1. Let a be the units digit of N; then clearly,

$$a^2 - a \equiv 0 \text{ (mod. } r) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

2, If r is a prime, the only possible solutions of (i) are $a = 0$ and $a = 1$; for a and $a-1$ are prime to each other and each is $< r$.

(α) If a is 0, the last two digits of N^2 are 00 and hence the last two digits of N are also 00; and so on clearly in this case all the digits of N are 0's.

(β) If a is 1, it is clear that if b is the tens digit in N, then the tens digit in N^2 is the least positive residue of $2b$, modulus r. Hence $2b \equiv b$ (mod. r).

$\therefore$ $b \equiv 0$; but $b < r$, therefore it is $= 0$; similarly, every other digit to the left is 0.

In other words, we get, in this case, the trivial solutions ......0000 and ......00001.

In what follows, we shall always reject these two solutions as being trivial.

3. If r is a power of a prime, it is obvious that the above argument still holds good.

Hence for scales whose radices are either primes or powers of primes there are no possible values of N. *E.g.*, if $r < 30$, there are no possible values of N for the scales 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29.

4. Next let $r = p^\lambda q^\mu$, where p and q are two different primes and λ, μ are any two positive integers. We may take $p > q$ without loss of generality.

We have in this case $a(a-1) \equiv 0 \text{ (mod } pq) \quad \ldots \quad \ldots \quad (2)$

This can be satisfied, since a and $a-1$ are prime to each other and p and q are also prime to each other; hence

(i) $a \equiv 0 \text{ (mod } p) = Pp$, say and $a - 1 = Qq$,

or (ii) $a - 1 = Pp$ and $a = Qq$.

Hence we must have $Pp - Qp = \pm 1. \quad \ldots \quad \ldots \quad (3)$

Let p/q be converted to a simple contd. fraction and let u/v be the penultimate convergent. Then one or other of the equations (3) is satisfied by $P=v$ and $Q=u$, since $pv-qu=\pm 1$. This gives for a the value pv which is $< pq$ (since $v < q$), ti e., $< r$, and therefore a possible digit.

Again, if $pv - qu = +1$, then $p(q-v) - q(p-u) = -1$.

Hence in this case there are always two possible solutions and only two.

5. The following table gives the values of a for given values of r:—

r = 6; 10; 12; 14; 15; 18; 20; 21; 22; 24; 26; 28.

a = 3, 4; 5, 6; 4, 9; 7, 8; 6, *t*; 9, *t*; 5, *s*; 7, *fi*; *e*, *tw*; 9, *s*; *th*, *f*; 8, *twe*;

where *t*, *e*, *tw*, *th*, *f*, *fi*, *s*, *twes*, stand respectively for ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen and twenty-one.

It will be observed that the sum of the two values of a is always $r+1$.

6. When the units digit is fixed, the other digits are determined uniquely. For, suppose d digits of N have been determined and let the number formed by these be denoted by N_d. Let x be the $(d+1)$th digit of N. Then the number formed with these $(d+1)$ digits is $x \cdot 10^d + N_d$. Its square is $x^2 10^{2d} + 2x \cdot 10^d \cdot N_d + N_d^2$, If x be the $(d+1)$th digit of this, it is also the $(d+1)$th digit of $2xN_d \cdot 10^d + N_d^2$. If δ be the $(d+1)$th digit of N_d^2 and a the first digit of N_d, it is clear that the $(d+1)$th digit is the least positive residue of $[2ax + \delta]$ (mod r). Hence x is to be determined from

$$2ax + \delta \equiv x \pmod{r}, \text{ or } \quad (2a-1)x \equiv 0.$$

This congruence has only one solution $< r$. For if x' were another solution, we would have

$$(2a-1)\,x \equiv (2a-1)x', \text{ i e., } (2a-1)\,(x-x') \equiv 0.$$

Now every factor of r divides either a or $a-1$ by virtue of (1) and if any factor were to divide also $2a-1 = a + (a-1)$, such factor would have to divide both a and $a-1$ which is impossible, since these are prime to each other. Hence $2a-1$ is prime to r. Therefore, $x - x' \equiv 0$; hence $x = x'$, if both are less than r.

Thus, if d digits of such a number have been determined, the $(d+1]$th digit is uniquely determined. In other words, the units digit determines the second, the first two the third, and so on.

Ex. Let $r=12$; then $a=4$ or 9. Let us take $a=4$. Then we get, step by step, the following table :—

d	N	N_d^2	δ	Congruence	x
1	4	14	1	$7x+1\equiv 0$	5
2	54	2454	4	$7x+4\equiv 0$	8
3	854	5e3854	3	$7x+3\equiv 0$	3
4	3854	...73854	7	$7x+7\equiv 0$	e
5	e3854	...5e3854	5	$7x+5\equiv 0$	1
6	1e3854	...61e3854	6	$7x+6\equiv 0$	6

and so on.

7. Another method of finding N is as follows :—

Find x and y to that $p^{\lambda n}x - q^{\mu n}y=1$.

Then, $N = p^{\lambda n} x$ is a solution.

For, then $N-1 = q^{\mu n} y$.

$$\therefore \quad N(N-1) = p^{\lambda n} q^{\mu n} xy = r^n xy \equiv 0 \pmod{r^n}.$$

Hence (N^2-N) ends in n zeros in the scale of r; *i.e.*, N^2 and N end in the same n digits in the scale of r.

Ex. 1.—Let $n=8$, $r=10$, so that $p=5$, $q=2$, $\lambda=\mu=1$.

We have to solve $5^8x-2^8y=1$. We get $x=33$, $y=50354$. Hence $N=12890625$.

Ex. 2.—Let $n=10$, $r=6$.

We have to solve $3^{10}x - 2^{10}y = 1$ (scale 10).

We get $x = 409$ and $y = 23585$ (scale 10).

$\therefore$ $N = 24151041$ (scale 10).
$= 2221350213$ (scale 6).

8. The following table gives the values of N for certain values of r:—

r	a	N
6	3	...2221350213
	4	...3334205344
10	5	...59918212890625
	6	...40081787109376
12	4	...21e61e3854
	9	...9t05t08369

It will be observed that the sum of the two values of N is always 1000......001.

N. B. Mitra.

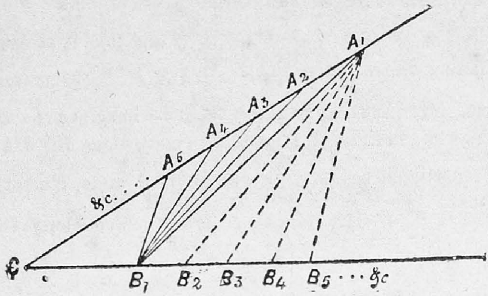
Geometrical Representation of an H. P.

The following is a simple way of representing a Harmonic Series :

We know that

$$\triangle ABC = \tfrac{1}{2}\, ab \sin C.$$

Supposing $\triangle$ and C are kept constant, and b increases in A.P., then a will decrease in H. P. and *vice versa.* This suggests to us the following figure :—



Let CA_1B_1 be a Δ.

Take $A_2A_3A_4$ ...... such that OA_1, OA_2, OA_3......, *i.e.* b_1, b_2, b_3 ...... are in A. P. $[A_1A_2 = A_2A_3 = A_3A_4 = ...]$.

Join B_1 to all the 'A's.

Then, draw through A_1

$$A_1B_2 \parallel B_1A_2$$
$$A_1B_3 \parallel B_2A_3$$
$$\cdots\cdots\cdots\cdots$$
$$A_1B_r \parallel B_1A_r.$$

Then will

OB_1, OB_2, OB_3, ... *i.e.*, a_1, a_2, a_3 ...... be in H. P.

G. S. Mahajani.

SOLUTIONS.

Question 1127.

(K. J. SANJANA, M.A.) :—Prove that there are two and only two Tucker circles of a triangle which touch a given straight line. These circles coalesce when the given line is one of the sides of the triangle.

If O and K be the circumcentre and symmedian point of a triangle ABC, T_1 the centre and R_1 the length of the radius of the Tucker circle touching BC, prove that

$$KT_1 : T_1O = b^2 + c^2 - a^2 : b^2 + c^2 + a^2 \text{ and } R_1 : R = bc : (b^2 + c^2)$$

Solution by Martyn M. Thomas and L. N. Subramaniam.

[Lemma. (1) Let three points D, E, F be imagined to start simultaneously from the vertices B, C, A and to move along BC, CA, AB with velocities proportional to $\frac{c}{b}$, $\frac{a}{c}$, $\frac{b}{a}$; and at the same instant, let three other points D′, E′, F′ start from C, A, B and move along CB, AC, BA with velocities bearing the same proportion to $\frac{b}{c}$, $\frac{c}{a}$, $\frac{a}{b}$. Then the points D, D′, E, E′, F and F′ will at any time determine one position of a Tucker circle. (2) The radius of a Tucker circle is capable of any value between R sin ω and ∞, the minimum Tucker circle being the auxiliary circle of the Brocard ellipse. (3) The centres of all the Tucker circles lie on the line joining the Lemoine Point and circumcentre of ABC].

From the above facts, it is clear that a given straight line in the plane of the triangle can be touched by only two Tucker circles.

From Lemma (1), it is seen that the points D and D′ will gradually move up to and cross each other. If the triangle is scalene, the times of coincidence of D with D′, and of E with E′, and of F with F′, will all be different.

To find the position of coincidence for the points D, D′ (in which case the corresponding Tucker circle touches the side BC), observing that BD, CD′ are always proportional to c/b, b/c, we have, at the instant of coincidence, $BD : DC = c/b : b/c = c^2 : b^2$.

Hence D lies at the foot of the symmedian drawn through A, which is therefore the point of contact of the Tucker circle touching BC.

Again, for any Tucker circle, round DEF,

$$\frac{\text{circum-radius of DEF}}{\text{circum-radius of ABC}} = \text{ratio of similitude } \frac{\Omega F}{\Omega A} = \frac{\sin \omega}{\sin(\theta + \omega)},$$

where Ω is the positive Brocard point.

If the Tucker circle DEF touches the side BC,

$$\text{angle EFD} = \text{angle EDC} = \theta.$$

Hence, since $\text{EFB} = \text{FAE} + \text{FEA}$, $2\theta = \text{A} + \theta$.

Thus $\theta = \text{A}$, which is also seen to be true from the fact that, D being in this case the foot of the symmedian from A, DE is anti-parallel to AB

$$\therefore \quad \frac{\text{R}_1}{\text{R}} = \frac{\sin\omega}{\sin(\text{A}+\omega)} = \frac{1}{\sin\text{A}\cot\omega + \cos\text{A}}$$

$$= \frac{1}{\sin\text{A}\,.\,\dfrac{a^2+b^2+c^2}{4\Delta} + \cos\text{A}}$$

$$= \frac{1}{\sin\text{A}\,.\,\dfrac{a^2+b^2+c^2}{2bc\sin\text{A}} + \dfrac{b^2+c^2-a^2}{2bc}} = \frac{bc}{b^2+c^2}.$$

Lastly: if T_1 is the centre of the Tucker circle touching BC, T_1 lies on OK and the perpendicular T_1M on BC is equal to the radius R_1 found above.

$$\therefore \quad \frac{\text{KT}_1}{\text{T}_1\text{O}} = \frac{\text{KL} - \text{T}_1\text{M}}{\text{T}_1\text{M} - \text{ON}},$$ where KL, ON are perpendiculars on BC,

$$= \frac{\frac{1}{2}a\tan\omega - \dfrac{\text{R}\sin\omega}{\sin(\text{A}+\omega)}}{\dfrac{\text{R}\sin\omega}{\sin(\text{A}+\omega)} - \text{R}\cos\text{A}}$$

$$= \frac{\sin\text{A}\sin\omega\sin(\text{A}+\omega) - \cos\omega\sin\omega}{\cos\omega\,[\sin\omega - \cos\text{A}\sin(\text{A}+\omega)]}$$

$$= \tan\omega\,.\,\frac{\sin\text{A}\sin(\text{A}+\omega) - \cos(\overline{\text{A}+\omega}-\text{A})}{\sin(\overline{\text{A}+\omega}-\text{A}) - \cos\text{A}\sin(\text{A}+\omega)}$$

$$= \tan\omega\,.\,\frac{\cos\text{A}\cos(\text{A}+\omega)}{\cos(\text{A}+\omega)\sin\text{A}}$$

$$= \frac{4\Delta}{a^2+b^2+c^2}\,.\,\frac{b^2+c^2-a^2}{2bc\sin\text{A}} = \frac{b^2+c^2-a^2}{a^2+b^2+c^2}.$$

Question 1128.

(K. J. Sanjana and M. K. Kewalramani):—If ${}_r\text{S}_p$ = the sum of the roducts p at a time of the first r natural numbers (${}_2\text{S}_0 = 1$), show that

$$(1) \quad \frac{{}_2\text{S}_0}{3\,!} - \frac{{}_4\text{S}_2}{5\,!} + \frac{{}_6\text{S}_4}{7\,!} - \dots\dots = \frac{\pi}{96}\left[\frac{\pi^2}{4} - 3(\log 2)^2\right].$$

$$(2) \quad \frac{{}_3\text{S}_1}{4\,!} - \frac{{}_5\text{S}_3}{6\,!} + \frac{{}_7\text{S}_5}{8\,!} - \dots\dots = \frac{\pi^2}{64}\log 2 - \frac{1}{48}(\log 2)^3.$$

Solution by Martin M. Thomas, and several others.

Now $$(1+x)^y \equiv e^{y \log (1+x)}.$$

Equating the co-efficient of y^3 on both sides

$$\frac{[\log (1+x)]^3}{3!} = \text{coefft. of } y^3 \text{ in } \left[1 + yx + \frac{y(y-1)}{2!}x^2 + \dots\right]$$

$$= \frac{x^3}{3!} - \frac{{}_3S_1}{4!}x^4 + \frac{{}_4S_2}{5!}x^5 - \frac{{}_5S_3}{6!}x^6 + \dots$$

Putting $x = i$,

$$\frac{[\log (1+i)]^3}{6} = -i\left\{\frac{{}_2S_0}{3!} - \frac{{}_4S_2}{5!} + \frac{{}_6S_4}{7!} - \frac{{}_8S_6}{9!} + \dots\dots\right\}$$

$$-\left\{\frac{{}_3S_1}{4!} - \frac{{}_5S_3}{6!} + \frac{{}_7S_5}{8!} - \dots\dots\right\}.$$

$$\therefore \quad \frac{\left[\frac{1}{2}\log 2 + i \cdot \frac{\pi}{4}\right]^3}{6} = -i\,P - Q,$$

where P and Q are the given series.

Equating imaginary and real parts

$$P = \frac{\pi^3}{384} - \frac{\pi}{32}(\log 2)^2,$$

and $$Q = \frac{\pi^2}{64}\log 2 - \frac{1}{48}(\log 2)^3.$$

Question 1129.

(S. Malhari Rao):—Find a number which when multiplied by 2 or 3 or 4 or 5 or 6 or 7 or 8 or 9 gives in each case a product which contains the same figures as the number itself.

Solution by Mrs. Edith Thomas.

Now $$\frac{1}{17} = \dot{0}58823529411764\dot{7}.$$

Consider the number $N = \cdot 0588235294117647.$

Then

$$2N = \cdot 1176470588235294$$
$$3N = \cdot 1764705882352941$$
$$4N = \cdot 2352941176470588$$
$$5N = \cdot 2941176470588235$$
$$6N = \cdot 3529411764705882$$
$$7N = \cdot 4117647058823529$$
$$8N = \cdot 4705882352941176$$
$$9N = \cdot 5294117647058823,$$

all of which contain the same digits as the original number.

Hence the required number is ·0588235294117647.

Question 1142.

(K. ANANDA RAO) :—Let $\omega(n)$ and $\mu(n)$ be defined as follows :— $\omega(n) = 0$ if n has any prime factor repeated more than twice ; $\omega(n) = (-1)^r 2^r$ otherwise, r being the number of non-repeated prime factors of n ; $\mu(n) = 0$ if n has any repeated prime factors and $\mu(n) = (-1)^r$ otherwise, r having the same meaning as above. Show that for $|x| < 1$

$$\sum_1^\infty \mu(n)x^n = \sum_1^\infty \frac{\omega(n)\, x^n}{1 - x^n}.$$

Solution by G. A. Srinivasan.

This is a particular case of Ex. 24, p. 94 of Bromwich, *Infinite Series*.

Convert the series on the right-hand side into the double series,

$$\sum_{(m,\, n)} \omega(n)\, x^{n.m}$$

and re-arrange according to powers of x. The co-efficient of x^n will be

$$\lambda(n) \equiv \sum_d \omega(n)$$

the summation being extended to all divisors of n (including 1 and n).

Suppose $\quad n = p^\alpha q^\beta r^\gamma \ldots\ldots\ldots \quad (\alpha \geq 1,\ \beta \geq 1, \ldots\ldots)$
where p, q, r ... are different primes.

The divisors of n are the terms in the product

$$(1 + p + p^2 + \ldots + p^\alpha)(1 + q + q^2 + \ldots\ldots + q^\beta)(1 + r + \ldots\ldots r^\gamma)\ldots\ldots$$

and $\lambda(n)$ is the sum of the ω's taken for each separate term. By the definition of ω, we see that $\lambda(n)$ can be obtained by replacing in each term of the product each of the non-repeated primes by -2, each squared prime by 1 and higher powers of primes by 0.

Consequently

$$\lambda(n) = (1 - 2 + 1 + 0 + \ldots)(- - -)$$
$$= 0,$$

if n has any repeated prime factor

and $$\lambda(n) = (1-2)(1-2)\ldots\ldots\ldots$$
$$= (-1)^r,$$

if n has r prime factors no one being repeated.

Hence $\qquad \lambda(n) \equiv \mu(n).$

To justify the re-arrangement of the double series, we note that it is absolutely convergent when $| x | < 1$, on comparing it with

$$\Sigma 4 \sqrt[3]{n} \frac{x^n}{1-x^n}, \quad | \omega(n) | \leq 2^r$$

since

where r is the number of prime factors of n. But r is always less than the number of digits in n (in the decimal scale) plus 2,

so that
$$| \omega(n) | < 2^{\log_{10}(n)+2}$$
$$< 4\, n^{\log_{10} 2}$$
$$< 4 \sqrt[3]{n}.$$

Question 1145.

(N. Dorai Rajan) :—n rods OA_1, OA_2, ... OA_n are hinged together at O, which is a plane joint. Show that the area of the plane polygon A_1A_2 ...... A_n is a maximum, when the circles on A_1A_2, A_2A_3, ... A_nA_1 as diameters have a common orthogonal circle ; and that the perimeter is a maximum when all the sides touch a circle. When there are only 3 rods, can the triangle be constructed with the ruler and compasses ?

Solution by Martyn M. Thomas.

Let the fixed lengths OA_1, OA_2 ...... be r_1, r_2, ...
and the variable lengths A_1A_2, A_2A_3 ...... be $2a_1$, $2a_2$, ...
and the variable angles A_1OA_2, A_2OA_3 ...... be θ_1, θ_2, ...

Then
$$\theta_1 + \theta_2 + \ldots\ldots + \theta_n = 360^\circ,$$
so that
$$d\theta_1 + d\theta_2 + \ldots\ldots + d\theta_n = 0. \qquad \ldots \quad (1)$$

Area of the plane polygon A_1A_2 ...... A_n is given by

$$\Delta = \tfrac{1}{2} r_1r_2 \sin\theta_1 + \tfrac{1}{2} r_2r_3 \sin\theta_2 + \ldots\ldots + \tfrac{1}{2} r_nr_1 \sin\theta_n.$$

When Δ is a maximum, $d\Delta = 0$.

$$\therefore \quad r_1r_2 \cos\theta_1 \,.\, d\theta_1 + r_2r_3 \cos\theta_2 \,.\, d\theta_2 + \ldots\ldots$$
$$+ r_nr_1 \cos\theta_n \,.\, d\theta_n = 0, \qquad \ldots \quad (2)$$

subject to the relation (I).

$$\therefore \quad r_1r_2 \cos\theta_1 = r_2r_3 \cos\theta_2 = \ldots\ldots = r_nr_1 \cos\theta_n = k^2, \text{ say}$$

$$\therefore \quad r_1^2 + r_2^2 - 4a_1^2 = r_2^2 + r_3^2 - 4a_2^2 \ \ldots\ldots$$
$$= r_n^2 + r_1^2 - 4a_n^2 = 2k^2.$$

That is, $2(OM_1^2+a_1^2)-4a_1^2=2(OM_2^2+a_2^2)-4a_2^2=\ldots=2k$, where M_1, M_2 ...... are the mid. points of the sides A_1A_2, A_2A_3 ......

$$\therefore \quad OM_1^2-a_1^2=OM_2^2-a_2^2=\ldots\ldots=k^2.$$

Hence a circle with O as centre, and k as radius, will cut the circles on A_1A_2, A_2A_3 ...... as diameters, at right angles.

Again, the perimeter of the polygon A_1A_2 ...... A_n is given by

$$p=2a_1+2a_2+\ldots\ldots+2a_n.$$

When p is a maximum, $dp=0$.

$$\therefore \quad da_1+da_2+\ldots\ldots+da_n=0\ldots \quad \ldots \quad \ldots \quad (3)$$

Since $\quad 4a_1^2=r_1^2+r_2^2-2r_1r_2\cos\theta_1$,

$$8a_1\,da_1=2r_1r_2\sin\theta_1\,d\theta_1,$$

$$\therefore \quad da_1=\frac{r_1r_2\sin\theta_1}{4a_1}d\theta_1=\frac{2p_1a_1}{4a_1}d\theta_1=\tfrac{1}{2}p_1d\theta_1,$$

where p_1 is the perpendicular from O on the side $2a_1$.

Similarly, $\quad da_2=\frac{1}{2}p_2d\theta_2$, $da_3=\frac{1}{2}p_3d\theta_3$, etc.

Thus relation (3) becomes

$$p_1d\theta_1+p_2d\theta_2+p_3d\theta_3+\ldots\ldots+p_nd\theta_n=0,$$

subject to the relation (1) found above.

$$\therefore \quad p_1=p_2=p_3=\ldots\ldots=p_n=l, \text{ say.}$$

Hence a circle with O as a centre and radius l will be touched by the sides A_1A_2, A_2A_3 ...... of the polygon.

Lastly when there are only 3 rods, the relation written immediately after the equation (2), shows that O will be the orthocentre of the triangle $A_1A_2A_3$ in the first case. In the second case, O is clearly the incentre of $A_1A_2A_3$.

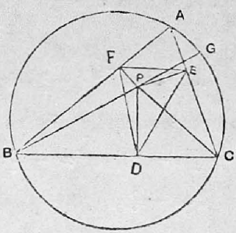
Hence the construction required is :—Given three lengths OA, OB, OC in magnitude ; find their positions so that (1) O is the orthocentre of ABC ; and (2) O is the incentre of ABC.

Question 1149.

(S. Anantarama Aiyar) :—P is a variable point in the plane of a △ABC which moves so that, if PD, PE, PF are drawn perpendicular to the sides, DEF remains constant in area. Prove geometrically that P moves on a circle.

Solution by N. Sundaram Aiyar and several others

Let BP meet circle ABC at G.



$$\triangle \text{FDE} = \tfrac{1}{2}\,\text{FD}\,.\,\text{DE} \sin \text{F}\hat{\text{D}}\text{E}$$

$$= \tfrac{1}{2}\,(\text{PB} \sin \text{B})\,(\text{PC} \sin \text{C}) \sin \text{F}\hat{\text{D}}\text{E}.$$

$$\text{But F}\hat{\text{D}}\text{E} = \text{F}\hat{\text{P}}\text{E} - \text{P}\hat{\text{F}}\text{D} - \text{P}\hat{\text{E}}\text{D}$$

$$= \pi - \text{A} - \text{PBC} - \text{PCD}$$

$$= \pi - \text{BGC} - \text{GPC}$$

$$= \text{PCG}.$$

$$\therefore\ \triangle\text{FDE} = \tfrac{1}{2}\,\text{PB} \sin \text{B} \sin \text{C}\,.\,\text{PC} \sin \text{P}\hat{\text{C}}\text{G}$$

$$= \tfrac{1}{2}\,\text{PB} \sin \text{B} \sin \text{C}\,.\,\text{PG} \sin \text{P}\hat{\text{G}}\text{C}$$

$$= \tfrac{1}{2} \sin \text{B} \sin \text{C} \sin \text{A}\,.\,\text{PG}\,.\,\text{PB}$$

$= \tfrac{1}{2} \sin \text{A} \sin \text{B} \sin \text{C}\,(\text{R}^2 - \text{OP}^2)$, if O be the circum-centre and R the circum-radius.

Hence if $\triangle$FDE is to have a constant area, OP is to be constant, or the locus of P is a circle concentric with the circum-circle.

Question 1152.

(Selected) :—Find the complete primitive of the differential equation

$$9xy^2y_2 - 2 = 0. \quad \text{(Forsyth : } \textit{Diff. Eqns.}\text{)}$$

Remarks by Prof. Sanjana.

The equation given by Dr. Forsyth is $9xy^2\dfrac{d^2y}{dx^2} + 2 = 0$; he obtains the intermediate solutions

$$y^{\frac{3}{2}} - \text{A}x^{\frac{3}{2}} = x^{\frac{1}{2}},\ y^{\frac{3}{2}} - \text{B}x^{\frac{3}{2}} = -x^{\frac{1}{2}},$$

and says that these equations do not coexist and thinks that a primitive infinite form exists.

As $y = x^{\frac{1}{3}}$ is an exact solution, if we put $y = x^{\frac{1}{3}}v$, v must be of the form $1 + kx^{\alpha} + \ldots\ldots$, where α is the lowest index. Making this substitution, we get

$$2 + 9x.x^{\frac{2}{3}}(1 + 2kx^{\alpha} \ldots.) \times x^{-\frac{5}{3}}\{-\tfrac{2}{9} + k(\alpha + \tfrac{1}{3})(\alpha - \tfrac{2}{3})\,x^{\alpha}\} = 0.$$

The constant terms cancel out; and, for x^{α} to disappear, we must have

$$\tfrac{4}{9}k = k(\alpha^2 - \tfrac{1}{3}\alpha - \tfrac{2}{9}), \text{ or } \alpha^2 - \tfrac{1}{3}\alpha - \tfrac{2}{3} = 0.$$

Thus α may be either 1 or $-\frac{2}{3}$.

Hence setting $y = x^{\frac{1}{3}}(1 + ax + bx^2 + cx^2 + \ldots\ldots)$, and making the necessary substitutions, we get

$$9x^{\frac{5}{3}}\{1 + 2ax + (a^2 + 2b)x^2 + \ldots.\} \times x^{-\frac{5}{3}}\{-\tfrac{2}{9} + \tfrac{4}{9}ax \times \tfrac{28}{9}bx^2 \ldots..\} + 2 = 0.$$

The constant and the lowest power of x disappear; the co-efficient of x^2 is $6a^2 + 24b$ on the left, giving $b = -\frac{1}{4}a^2$. So also c, d, ... may be found in terms of a.

Again setting

$$y = x^{\frac{1}{3}}(1 + Ax^{-\frac{2}{3}} + Bx^{-\frac{4}{3}} + \ldots\ldots),$$

and proceeding similarly, we get $B = -\frac{3}{7}A^2$; and C, D ...... may be similarly found.

Hence the primitive may be put in the form

$$y = x^{\frac{1}{3}}(\tfrac{1}{2} + ax - \tfrac{1}{4}a^2x^2 \ldots\ldots)$$
$$+ x^{\frac{1}{3}}(\tfrac{1}{2} + Ax^{-\frac{2}{3}} - \tfrac{3}{7}A^2x^{-\frac{4}{3}} \ldots\ldots),$$

which involves the two constants required by the order of the equation.

If we put $y = x^{\frac{1}{3}}(1 + v)$, the resulting equation is equally intractable. In the question here proposed, we have -2 for $+2$, so that we should start with $y = -x^{\frac{1}{3}}$.

Question 1155

(R. Vaidyanathaswami) :—ABCE are four points on a conic S. The polar line of ABC w. r. t. S intersects S in P, Q and the polar line of E w. r. t. ΔABC intersects S in X, Y. Prove that the conic which is inscribed to ABC and XYP passes through E.

Solution by Hemraj and others.

If ABC be taken as the triangle of reference and E be (f, g, h), then

$$S \equiv \frac{U}{\alpha} + \frac{V}{\beta} + \frac{W}{\gamma} = 0.$$

The polar line of E w. r. t. ABC is

$$\frac{\alpha}{f} + \frac{\beta}{g} + \frac{\gamma}{h} = 0 \ldots \qquad \ldots \qquad \ldots \quad (1)$$

which cuts S in X $(\alpha_1, \beta_1, \gamma_1)$ and Y $(\alpha_2, \beta_2, \gamma_2)$,

The equation of XY is

$$\frac{U\alpha}{\alpha_1\alpha_2} + \frac{V\beta}{\beta_1\beta_2} + \frac{W\gamma}{\gamma_1\gamma_2} = 0 \qquad \ldots \qquad \ldots \quad (2)$$

Comparing (1) and (2) we get

$$\frac{Uf}{\alpha_1\alpha_2} = \frac{Vg}{\beta_1\beta_2} = \frac{Wh}{\gamma_1\gamma_2} \qquad \ldots \qquad \ldots \quad (3)$$

The equation of the conic touching the sides of ABC is

$$\Sigma \equiv \sqrt{\lambda\alpha} + \sqrt{\mu\beta} + \sqrt{\upsilon\gamma} = 0.$$

The polar line of ΔABC w. r. t. S is

$$\frac{\alpha}{U} + \frac{\beta}{V} + \frac{\gamma}{W} = 0,$$

which cuts S in P (f_1, g_1, h_1) and Q (f_2, g_2, h_2).

$$\therefore \quad \frac{f_1}{U} + \frac{g_1}{V} + \frac{h_1}{W} = 0. \qquad \ldots \qquad \ldots \quad (4)$$

XY, YP, PX touch Σ, if

$$\lambda f + \mu g + \upsilon h = 0 \ldots \qquad \ldots \qquad \ldots \quad (5)$$

$$\left.\begin{aligned} \frac{\lambda f f_1}{\alpha_1} + \frac{\mu g g_1}{\beta_1} + \frac{\upsilon h h_1}{\gamma_1} &= 0, \\ \frac{\lambda f f_1}{\alpha_2} + \frac{\mu g g_1}{\beta_2} + \frac{\upsilon h h_1}{\gamma_2} &= 0. \end{aligned}\right\} \text{ by (3).}$$

From the last two equations, we get

$$\frac{\lambda f f_1}{U} = \frac{\mu g g_1}{V} = \frac{\upsilon h h_1}{W}. \qquad \ldots \qquad \ldots \quad (6)$$

$\therefore$ from (4) and (6), we have

$$\frac{1}{\lambda f} + \frac{1}{\mu g} + \frac{1}{\upsilon h} = 0. \qquad \ldots \qquad \ldots \quad (7)$$

(6) and (7) give $\sqrt{\lambda f} + \sqrt{\mu g} + \sqrt{\upsilon h} = 0$, which shows that Σ passes through E.

Question 1158.

(N. DORAIRAJAN) :—If the sides of a triangle are in arithmetical progression, prove that, with the usual notation, the straight line IG is parallel to one of the sides.

Solution by N. Sundaram Aiyar, C. N. Sreenivasa Iyengar and N. Narayanaswami.

Let a, b, c the sides of the $\triangle$ ABC be in A P. Then

$$a + c = 2b, \text{ or } a + b + c = 3b. \quad \ldots \quad \ldots \quad (1)$$

If S be the area of the triangle, r the in-radius, s the semi-perimeter, p the altitude of the triangle from B, and IL and GM be the perpendiculars to CA, we have

$$\text{GM} = \tfrac{1}{3} p = \tfrac{1}{3} \cdot \frac{2\text{S}}{b} = \frac{2\text{S}}{a + b + c} = r = \text{IL}.$$

Hence IG is parallel to ML or to CA.

Similar Solution by G. S. Mahajani and several others.

Question 1160.

(N. DORAIRAJAN) :—A number N consists of p digits. It has the property that when squared, it ends with the same digits in same order as in itself. Find the number when (1) $p = 4$ and (2) $p = 5$.

Solution by N. B. Mitra, and Hemraj.

Let the digits in N commencing from the unit's place be a, b, c, d......, so that

$$\text{N} = a + 10b + 10^2c + 10^3d + \ldots\ldots$$

$$\therefore \ \text{N}^2 = a^2 + 2ab \cdot 10 + (b^2 + 2ac)\, 10^2 + \ldots\ldots$$

Hence, by the question, $a \equiv a^2 \pmod{10}$ ... ... (1)

$$a + 10b \equiv a^2 + 20ab \pmod{10^2} \quad \ldots \quad \ldots \quad (2)$$

$$a + 10b + 10^2c \equiv a^2 + 20ab + 10^2\,(b^2 + 2ac) \pmod{10^3} \quad (3)$$

$$a + 10b + 10^2c + 10^3d \equiv a^2 + 20ab + 10^2\,(b^2 + 2ac) + 10^3\,(2ad + 2bc) \pmod{10^4} \quad \ldots \quad (4)$$

$$a + 10b + 10^2c + 10^3d + 10^4e \equiv a^2 + 20ab + 10^2\,(b^2 + 2ac) + 10^3\,(2ad + 2bc) + 10^4\,(c^2 + 2bd + 2ac) \pmod{10^5} \quad \ldots \quad \ldots \quad (5)$$

&c., &c., &c.

From (1) $a = 1$, 5 or 6 (zero being excluded, as a trivial solution).

(i) First: let $a = 1$; then from (2)

$$10b \equiv 20b \pmod{10^2}, \text{ or } b \equiv 2b \text{ whence } b = 0.$$

$\therefore$ from (3) $10^2c \equiv 10^2 . 2c \pmod{10^3}$, whence $c = 0$.

It will be found similarly that in this case d, e, f ... are all zero.

Hence we get the trivial solution N = ... 0000001

(ii) Secondly let $a = 5$. We then have from (2),

$$5 + 10b \equiv 25 + 100b \pmod{10^2}$$

or $\quad 10b \equiv 20 \pmod{10^2}$, or $b \equiv 2 \pmod{10}$; whence $b = 2$.

From (3) $\quad 100c + 25 \equiv 1000c + 625 \pmod{10^3}$

or $\quad 100c \equiv 600.$,,

$\therefore \quad c = 6.$

Substituting in (4), $10^3d + 625 \equiv 10^4d + 3{\cdot}625 \pmod{10^4}$

or $\quad 10^3d \equiv 0 \pmod{10^4}$.

$\therefore \quad d = 0.$

Substituting in (5), $10^4e + 625 \equiv 390625 + 10^5e \pmod{10^5}$

or $\quad 10^4e \equiv 90000$,,

$\therefore \quad e = 9.$

And so on. We get successively $f = 8$, $g = 2$, $h = 1$.

(iii) Finally, let $a = 6$. Then from (2)

$$(10b + 6) \equiv 120b + 36 \pmod{10^2}$$

or $\quad 10b \equiv -30$,,

$$\therefore b = 7.$$

From (3) $\quad 100c + 76 \equiv 1200c + 5776 \pmod{10^3}$

or $\quad 100c \equiv -700$,,

or $\quad c \equiv -7 \pmod{10}$

$\therefore \quad c = 3.$

Hence from (4),

$$1000d + 376 \equiv 12000d + 51376 \pmod{10^4}$$

$$1000d \equiv -1000 \quad ,,$$

$$d \equiv -1 \pmod{10}$$

$$\therefore \quad d = 9.$$

From (5) $\quad 10^4e + 9376 \equiv 120000e + 1509376 \pmod{10^5}$

or $\quad 10^4e \equiv 2 . 10^4e$,,

$\therefore \quad e = 0.$

Hence, we see that the only solutions are

for $p = 4$, $\quad$ N = 9376,

for $p = 5$, $\quad$ N = 90625.

N. B. The following solutions for other values of p may be noted :—

p	N
1	1,5,6
2	25,76
3	376,625
6	109376 ; 890625
7	7109376 ; 2890625
8	87109376 ; 12890625
9	787109376 ; 212890625
10	1787109376 ; 8212890625
11	81787109376 ; 18212890625
12	918212890625
13	9918212890625
14	40081787109376 ; 59918212890625

Also partial solutions on similar lines by C. N. Sreenivasa Iyengar and K. Sreenivasa Raghavan.

Question 1161.

(K. J. Sanjana) :—Prove that

$$\frac{n(n+1)\ldots(2n-1)}{n\,!}+\frac{n(n+1)\ldots 2n}{(n+1)\,!}\cdot\frac{1}{2}+\frac{n(n+1)\ldots(2n+1)}{(n+2)}\cdot\frac{1}{2^2}+\ldots\ ad\ inf.=2^{2n-1}.$$

Solution by B. B. Bagi and G. V. Gurjar and N. B. Mitra.

If ${}_pH_r$ denotes the number of homogenous products of p letters of r dimensions, then ${}_pH_r = {}_{p+1}H_r - {}_{p+1}H_{r-1}$ so that the required sum

$$S(n) = {}_nH_n + {}_nH_{n+1}\,\frac{1}{2} + {}_nH_{n+2}\cdot\frac{1}{2^2} + {}_nH_{n+3}\cdot\frac{1}{2^3} + \ldots$$

$$= ({}_{n+1}H_n - {}_{n+1}H_{n-1}) + ({}_{n+1}H_{n+1} - {}_{n+1}H_n)\,\frac{1}{2}$$

$$+ ({}_{n+1}H_{n+2} - {}_{n+1}H_{n+1})\,\frac{1}{2^2} + \ldots$$

$$= {}_{n+1}H_{n+1}\cdot\frac{1}{2^2} + {}_{n+1}H_{n+2}\cdot\frac{1}{2^3} + \ldots$$

$$= \frac{1}{2^2}\,S\,(n+1).$$

So, if $S(n)=2^{2n-1}$, then $S(n+1) = 2^{2n+1}$.

Obviously $S(1)=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots = 2.$

Hence, by induction, the result is true generally.

Question 1167.

(HEMRAJ) :—If $I \equiv (ac-4bd+3c^2)$ is a negative number, prove that a necessary and sufficient condition that the roots of the biquadratic

$$ax^4 + 4bx^3 + 6cx^2 + 4da + e = 0,$$

with real co-efficients, may be concyclic in the Argand Diagram is that a root of the reducing cubic should be proportional to $\sqrt{I}$.

Solution by K. J. Sanjana.

(1) As I is negative, $\Delta = I^3 - 37J^2$ is also negative and the biquadratic has two real and two imaginary roots. Let these roots, α, β, γ, δ, be denoted by the numbers x_1, x_2, $x_3 + ix_4$, $x_3 - ix_4$ respectively. Then for the four corresponding points in the Argand Diagram to be concyclic,

we must have $$(x_3-x_1)(x_2-x_3)=x_4^2,$$

or $$x_2x_3+x_3x_1-x_1x_2=x_3^2+x_4^2.$$

Substituting for x_1, x_2, x_3, x_4, their values in terms of the roots,

we get $$2(\alpha\beta + \gamma\delta) - (\alpha+\beta)(\gamma + \delta) = 0,$$

so that $$(\beta-\gamma)(\alpha - \delta) - (\gamma - \alpha)(\beta - \delta) = 0.$$

But the last expression is 12 times one root of the reducing cubic. Hence in this case $J = 0$, and the cubic becomes $4a^3\theta^3 - I\alpha\,\theta = 0$ giving for the other roots $2\alpha\theta = \pm\sqrt{I}$. It is thus necessary that a root of this cubic should vary as $\sqrt{I}$ to ensure a cyclic quadrilateral.

(2) Again, let one root of the reducing cubic be $k\sqrt{I}$; then a second root will be $-k\sqrt{I}$, and let the remaining root be denoted by ω. The cubic then becomes $(\theta^2-k^2I)(\theta-\omega) = 0$, or $\theta^3-\omega\theta^2-k^2I\theta+k^2I\omega = 0$. Comparing this with the form $4a^3\theta^3-I\alpha\theta+J=0$, we get $\omega=0$, so that one of its roots, say $(\beta-\gamma)(\alpha-\delta) - (\gamma-\alpha)(\beta-\delta)$, vanishes. Hence $2(\alpha\beta+\gamma\delta) - (\alpha+\beta)(\gamma+\delta)=0$, and the number representing the roots form a cyclic quadrilateral. To ensure this, therefore, it is sufficient that one root of the cubic should be proportional to $\sqrt{I}$.

(c) If I is a positive number greater than $3J^{\frac{2}{3}}$, Δ is positive, and the roots will be all imaginary when H and $a^2I - 12H^2$ are also positive. In this case too, the roots will give a cyclic diagram, because the extremities of two parallel line-segments which have the same right bisector are concyclic.

Similar Solution by C. N. Sreenivasa Iyengar.

Questions 1168 and 1169.

(S. KRISHNASWAMI AIYANGAR) :—

(1) If (x, y) stand for $\text{sn}\,(x+y)\,\text{sn}\,(x-y)$, then

$$\frac{1-k(\alpha\beta)}{1+k(\alpha\beta)} \times \frac{1-k(\beta\gamma)}{1+k(\beta\gamma)} = \frac{1+k(\gamma\delta)}{1-k(\ \delta)} \times \frac{1+k(\delta\alpha)}{1-k(\delta\alpha)}$$

(2) If $(xy) \equiv \dfrac{dn^2x - dn^2y}{1+k^2dn^2x\,dn^2y}$, show that

$$(\alpha\beta) + (\beta\gamma) + (\gamma\alpha) = k^2(\alpha\beta)\,(\beta\gamma)\,(\gamma\alpha).$$

Solution by F. H. V. Gulasekharam.

(1) Let $\tanh A = k\,\text{sn}^2\,\alpha$; $\tanh B = k\,\text{sn}^2\,\beta$; $\tanh C = k\,\text{sn}^2\,\gamma$.

Then
$$k(\beta\gamma) = \tanh\,(B - C$$
$$k(\gamma\alpha) = \tanh\,(C - A)$$
$$k(\alpha\beta) = \tanh\,(A - B).$$

Now $\tanh\,(B-C) + \tanh\,(C-A) + \tanh\,(A-B)$
$+ \tanh\,(B-C)\tanh\,(C-A)\tanh\,(A-B) \equiv 0.$

Hence $$(\beta\gamma) + (\gamma\alpha) + (\alpha\beta) + k^2(\beta\gamma)\,(\gamma\alpha)\,(\alpha\beta) \equiv 0.$$

$$\therefore \quad \frac{1-k(\beta\gamma)}{1+k(\beta\gamma)}\cdot\frac{1-k(\gamma\alpha)}{1+k(\gamma\alpha)}\cdot\frac{1-k(\alpha\beta)}{1+k(\alpha\beta)} = 1.$$

$$\therefore \quad \frac{1-k(\alpha\beta)}{1+k(\alpha\beta)}\cdot\frac{1-k(\beta\gamma)}{1+k(\beta\gamma)} = \frac{1+k(\gamma\alpha)}{1-k(\gamma\alpha)} \quad \ldots \quad \ldots \quad (1)$$

Similarly

$$\frac{1-k(\gamma\delta)}{1+k(\gamma\delta)}\cdot\frac{1-k(\delta\alpha)}{1+k(\delta\alpha)} = \frac{1+k(\alpha\gamma)}{1-k(\gamma\alpha)} = \frac{1-k(\gamma\alpha)}{1+k(\gamma\alpha)} \quad \ldots \quad (2)$$

Hence from (1) and (2), the result follows immediately.

(2) Let $\tan A = k\,\text{dn}^2\,\alpha$; $\tan B = k\,\text{dn}^2\,\beta$; $\tan C = k\,\text{dn}^2\gamma$.

$$\therefore \quad k(\beta\gamma) = \tan\,(B - C)$$
$$k(\gamma\alpha) = \tan\,(C - A)$$
$$k(\alpha\beta) = \tan\,(A - B)$$

Now $\tan\,(B-C) + \tan\,(C-A) + \tan\,(A-B)$
$\equiv \tan\,(B-C)\,.\,\tan\,(C-A)\,.\,\tan\,(A-B).$

$$\therefore \quad (\beta\gamma) + (\gamma\alpha) + (\alpha\beta) \equiv k^2(\beta\gamma)\,(\gamma\alpha)\,(\alpha\beta).$$

[A large number of similar results could be immediately got, as in Greenhill's *Elliptic Functions*, by successively changing tan A and tanh A into $k\,\text{sn}^2\,\alpha$, $k\,\text{cn}^2\,\alpha$ and $k\,\text{dn}^2\,\alpha$ respectively.]

Question 1170.

(P. V. Seshu Aiyar):—If y_r is the rth term in the expansion of $(p+q)^n$, show that

$$\sum_{r=1}^{n+1}(y_r \cdot r_s) = \frac{d}{d}\left(\frac{q}{dq}d\right)\left(q\frac{d}{dq}\right)\ldots\ldots q(p+q)^n,$$

where $\frac{d}{dq}$ is repeated s times.

Solution by K. J. Sanjana and several others.

More generally, let $f(q)$ stand for any rational integral function of q —

$$a_1 + a_2 q + a_3 q^2 \ldots\ldots + a_n q^{n-1} + a_{n+1} q^n,$$

and let the operation of multiplying by q and then differentiating for q be denoted by $\left(\frac{d}{dq}\cdot q\right)$: then we shall have

$$\left(\frac{d}{dq}\cdot q\right)^s f(q) = \sum_{r=1}^{n+1}(r^s \cdot y_r).$$

For, $\left(\frac{d}{dq}\cdot q\right) f(q) = a_1 + 2a_2 q + 3a_3 q^2 \ldots + (n+1)a_{n+1}q^n$;

$$\left(\frac{d}{dq}\cdot q\right) f(q) = 1^2 \cdot a_1 + 2^2 a_2 q + 3^2 a_3 q^2 \ldots + (n+1)^2 a_{n+1} q^n;$$

and so on, the process being clearly capable of repetition. Thus

$$\left(\frac{d}{dq}\cdot q\right)^s f(q) = 1^s a_1 + 2^s a_2 q \ldots + (n+1)^s a_{n+1} q^n = \sum(r^s \cdot y_r).$$

Question 1179.

(F. H. V. Gulasekharam :—From an external point $T(x, y)$ tangents TP, TQ are drawn to the conic $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that the area of the triangle TPQ is

$$\frac{S^{\frac{3}{2}}\sqrt{-\Delta}}{\Delta - CS},$$

and that the area of the quadrilatral OPTQ, where O is the centre of the conic is $\frac{1}{2} \cdot \frac{\sqrt{-\Delta S}}{\Delta - CS}\left[x\frac{\partial S}{\partial x} + y\frac{\partial S}{\partial y}\right]$.

Solution by A. A. Krishnaswami Aiyangar and L. N. Subramanian.

The expression for the area of the quadrilateral is wrong. It ought to be merely $\frac{\sqrt{-\Delta S}}{C}$.

Changing the origin to T, the equation of the conic S may be written $ax^2 + 2hxy + by^2 + 2g'x + 2f'y + S = 0$. Denoting the centre of the conic by O, the co-ordinates of O are obviously $\left(\frac{G}{C}, \frac{F}{C}\right)$, where $G = hf' - bg'$, $F = g'h - af'$ and $C = ab - h^2$.

$$\therefore \qquad OT^2 = \frac{F^2 + G^2}{C^2} \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

The equation of PQ′ is $\qquad gx + f'y + S = 0,$

and that of OT is $\qquad \dfrac{x}{G} = \dfrac{y}{F}.$

The abscissæ x_1 and x_2 of the points P and Q can be easily seen to be the roots of the equation.

$x^2(Ff' + Gg') + 2xSG - SA = 0$, where $A = bS - f'^2$.

Now
$$PQ^2 = \frac{f'^2 + g'^2}{f'^2}(x_1 - x_2)$$
$$= \frac{4(f'^2 + g'^2)S}{f'^2(Ff'+Gg')^2}\{SG^2 + A(Ff' + Gg')\}.$$

Since the discriminant Δ of S is not altered by the change of origin

and since
$$Ff' + Gg' + SC = \Delta = \frac{AC - G^2}{b},$$
$$\frac{SG^2 + A(Ff' + Gg')}{f'^2} = \frac{(A - Sb)\Delta}{f'^2} = -\Delta.$$

$$\therefore \quad PQ^2 = \frac{4(f'^2 + g'^2)(-\Delta S)}{(Ff' + Gg')^2}. \qquad \ldots \qquad \ldots \text{(ii)}$$

Again, if θ be the angle between PQ and OT
$$\tan\theta = \frac{Ff' + Gg'}{Gf' - Fg'},$$
$$\therefore \quad \sin^2\theta = \frac{\tan^2\theta}{1 + \tan^2\theta} = \frac{(Ff' + Gg')^2}{(f'^2+g'^2)(F^2+G^2)} \quad \ldots \quad \ldots \text{(iii)}$$

From (i), (ii), and (iii), we deduce that the area of the quadrilateral
$$OPTQ = \tfrac{1}{2}\,OT\,.\,PQ\sin\theta$$
$$= \frac{\sqrt{-\Delta S}}{C}. \quad \ldots \quad \ldots \quad \ldots \text{(iv)}$$

If OT meets PQ at R, the co-ordinates of R are
$$\left\{\frac{-GS}{Ff' + Gg'}, \frac{-FS}{Ff' + Gg'}\right\}$$

and
$$\left|\frac{TR}{OT}\right| = \frac{CS}{Ff' + Gg'} = \frac{CS}{\Delta - CS}.$$

$\therefore$ The area of the triangle TPQ
$$= (\text{area of quadrilateral}) \times \left|\frac{TR}{OT}\right|$$
$$= \frac{\sqrt{-\Delta S}}{C}\cdot\frac{CS}{\Delta - CS} = \frac{S^{\frac{3}{2}}\sqrt{-\Delta}}{\Delta - CS} \quad \ldots \quad \ldots \text{(v)}$$

Cor. (1) Area of the triangle OPQ $= \dfrac{\Delta^{\frac{3}{2}}\sqrt{-S}}{C(\Delta - CS)}$. $\quad \ldots$ (vi)

(2) The chord PQ
$$= \frac{\sqrt{-\Delta S}}{\Delta - CS}\left\{\left|\frac{\partial S}{\partial x}\right|^2 + \left|\frac{\partial S}{\partial y}\right|^2\right\}^{\frac{1}{2}}. \quad \ldots \quad \ldots \text{(vii)}$$

QUESTIONS FOR SOLUTION.

1207. (G. V. T.) :—P is a point in the plane of a $\triangle$ ABC. X, Y, Z are the centres of circles PBC, PCA, PAB. If AX, BY, CZ concur at a point Q, the loci of P and Q are cubics whose equations in trilinear co-ordinates are

$$\sum \alpha(\beta^2 - \gamma^2)(\cos A \mp 2 \cos B \cos C) = 0,$$

the upper sign being for P and the lower for Q.

The former of these cubics cuts the circumcircle ABC in only one real point R (other than A, B, C) which is diametrically opposite to the Euler point of the $\triangle$ ABC. If L, M, N, be the reflexions of R in BC, CA, AB respectively, then AL, BM, CN meet in a point S. Show that RS is parallel to the Euler line.

1208. (G. V. T.) :—Given an angle of a $\triangle$ in position and the distance between the circum-centre and in-centre in magnitude and direction, construct the $\triangle$.

1209. (R. VAIDYANATHASWAMI) :—A one-one correspondence is established between the points on a conic S and the conics of a four-point system L. If the conics belonging to L are denoted by capitals and the corresponding points of S by small letters, shew that, if ABC are the line-pairs and E, S_1, S_2, S_3 four other conics of L, such that triangles could be inscribed in E with their sides touching S_1, S_2, S_3, respectively, then the conic which is inscribed in each of the triangles abc, $s_1 s_2 s_3$ passes through e.

1210. (R. VAIDYANATHASWAMI) :—Prove that in space of four dimensions (1) any three straight lines are met by one straight line, and that (2) there are five straight lines which meet each of six planes.

Are there any other non-trivial circumstances under which all the five transversals coalesce, except when (1) the six planes intersect two and two in three collinear points, (2) the six planes intersect two and two in three straight lines ?

If six planes have an infinite number of transversals, and if five of them are known, what is the locus of the sixth ?

1211. (A. NARASINGA RAO) :—Show how to construct a triangle having a given circle for its nine-points circle, and a given diameter of the circle as one of the pedal lines of the triangle.

Hence or otherwise determine all triangles which have a common pedal line system.

1212. (A. NARASINGA RAO):—Show that if the pedal lines of PQR with respect to the triangle ABC are concurrent, the pedal lines of ABC with respect to the triangle PQR are also concurrent.

1213. (V. RAMASWAMI AIYAR):—Prove that the pedal circle of any point P with respect to a triangle ABC cuts the side BC at an angle equal to the complement of the sum of the angles PAB, PBA, PCB,

1214. (G. V. TELANG):—If the centre of a pedal circle of a point P with respect to a triangle ABC lies on a line $l\alpha + m\beta + n\gamma = 0$, show that the locus of P is the cubic.

$$\Sigma (mc + nb)\, \alpha\, (\beta^2 + \gamma^2 + 2\beta\gamma \cos A) = 0$$

If the centre lies on the N. P. circle of the triangle ABC, the locus of the point P is $\sum \dfrac{a^3}{\alpha(\beta^2 + \gamma^2 + 2\beta\gamma\cos A)} = 0.$

1215. (F. H. V. GULASEKHARAM):—Shew that the trilinear equation to the pedal circle of a point (f, g, h) *w. r. t.* the triangle ABC may be expressed in the form:—

$$\begin{vmatrix} 0, & f/g + \cos C, & f/h + \cos B, & \alpha \\ g/f + \cos C, & 0, & g/h + \cos A, & \beta \\ h/f + \cos B, & h/g + \cos A, & 0, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} = 0.$$

1216. (F.H V. GULASEKHARAM):—The absolute normal co-ordinates of the centre of an inconic of the triangle of reference are α, β, γ. Prove that the squares of the semi-axes are given by the equation

$$a\sqrt{r^2 - \alpha^2} + b\sqrt{r^2 - \beta^2} + c\sqrt{r^2 - \gamma^2} = 0.$$

1217. (K. J. SANJANA):—(*a*) In a plane triangle ABC, if ω is the Brocard angle, prove that

$$\sin^2\omega\,\{\cot^3\omega + \cot(A-\omega)\cot(B-\omega)\cot(C-\omega)\}$$
$$= \operatorname{cosec} A \sin B \sin C + \operatorname{cosec} B \sin C \sin A + \operatorname{cosec} C \sin A \sin B.$$

(*b*) If G is the centroid, prove that

$$\cot GAB \cot GBC \cot GCA + \cot GAC \cot GCB \cot GBA = 6 \cot \omega.$$

1218. (B. VENKATESACHAR, M.A.):—Obtain an *asymptotic* series for the evaluation of

$$\phi(x) \equiv 1 + \frac{x}{1!} + \frac{x^2}{2!\sqrt{2}} + \frac{x^3}{3!\sqrt{3}} + \ldots \text{ ad inf.}$$

for large values of x, and find approximately the value of $\phi(-100)$.